Decompositions and extensions of operator valued representations of function algebras

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The present paper deals with some decompositions of operator representations of a quite arbitrary algebra $A \subset C(\Omega)$ with respect to the totality of all Gleason parts of A. Our approach is dilation free, that is no Ω-dilatability of the representation is required. The investigations which brought the author to the results enclosed in the present paper have been inspired by the question to what extend hold true the theorems of D. SARASON's paper [8]. In the light of the theorems which we present here the decomposition results enclosed in [8] appear to us as more subtle forms of some general decompositions. This is possible because of such special properties of the algebra A as the Dirichlet algebra property, the absence of non-zero completely singular orthogonal measures, the explicit characterization of some Gleason parts, the dilatability of the related representation, etc.

Let us point that some orthogonal decompositions appear in this paper as a consequence of a more general type of decomposition performed with the help of projections which are not necessarily self-adjoint. The basic means of our investigations is the abstract version of the M. and F. Riesz theorem given by I. GLICKSBERG in [3].

Section 3 concerns extensions of the absolutely continuous parts of representations to representations of some algebras of $\underset{\cdot}{H}^\infty$ type. Also in this case our methods are dilation free. Even more, we show that under the assumption of local character the related absolutely continuous representations are dilatable in a pretty way.

1. Throughout the present paper H stands for a complex Hilbert space with the inner product (x, y) $(x, y \in H)$ and the norm $\|x\| = \sqrt{(x, x)}$. The algebra of all linear bounded operators in H is denoted by $L(H)$. I is the identity operator in H. For $T \in L(H)$ $\|T\|$ is the norm and T^* is the adjoint of T. We say that $T \in L(H)$ is a projection if $T^2 = T$. The projection T is orthogonal if $T = T^*$.

Let Ω be a compact Hausdorff space and let $C(\Omega)$ and $C_R(\Omega)$ be the Banach algebra of all complex or real valued continuous functions on Ω, respectively, with the norm

$$\|u\| = \sup_{\Omega} |u(\omega)|.$$

The function algebra is by definition the subspace of $C(\Omega)$ which is closed under multiplication and contains constants. The functions belonging to the algebra are not required to separate the points of Ω.

We consider in the present paper merely Baire measures and Baire measurable functions. We say that the complex measure p is orthogonal to the algebra A and we write $p \perp A$ if $\int u\, dp = 0$ for all $u \in A$. We are now able to present some results of [3], which we need for our purposes.

Let φ be a homomorphism of A into the complex field. M_φ stands for the totality of all probability measures m on Ω such that

$$\int u\, dm = \varphi(u) \quad \text{for} \quad u \in A.$$

We say that a Baire set $\sigma \subset \Omega$ is φ-null if $m(\sigma) = 0$ for every $m \in M_\varphi$. The measure p is φ-absolutely continuous if it vanishes on every φ-null set. We then write $p \ll M_\varphi$. We say that the measure p is φ-singular if it is concentrated on a φ-null set. Every finite Baire measure has a unique φ-decomposition

(1.1) $$p = p^a + p^s$$

where $p^a \ll M_\varphi$ and p^s is φ-singular. The measure p is said to be completely singular if it is φ-singular for every φ.

The abstract M. and F. Riesz theorem proved in [3] reads as follows:

(1.2) If $p \perp A$ and (1.1) is the φ-decomposition of p, then $p^a \perp A$ and $p^s \perp A$.

It is proved moreover in [3] that the following holds true:

(1.3) If φ and ψ are in the same Gleason part G of the maximal ideals space $M(A)$ of A, then the φ-decomposition of p coincides with its ψ-decomposition. If φ and ψ are in different Gleason parts then the component $p^a \ll M_\varphi$ is ψ-singular.

It follows from the first part of (1.3) that p^a and p^s depend only on the Gleason part to which φ belongs. This justifies us to say that (1.1) is the decomposition of p with respect to the Gleason part $G(\varphi \in G)$ and write $p^a \ll G$ calling p^s the G-singular part of p.

Suppose we are given a linear map $T\colon A \to L(H)$ such that for some finite k

(1.4) $$\|T(u)\| \leqq k\|u\| \quad \text{for all} \quad u \in A.$$

This is a trivial consequence of the Hahn—Banach extension theorem and the Riesz representation theorem that there are measures $p_{x,y}$ $(x, y \in H)$ such that

(1.5) $$\|p_{x,y}\| \leqq k\|x\|\,\|y\| \quad \text{for} \quad x, y \in H,$$

(1.6) $$(T(u)x, y) = \int u\, dp_{x,y} \quad \text{for} \quad u \in A \quad \text{and} \quad x, y \in H.$$

The measures $p_{x,y}$ which satisfy (1.5) and (1.6) will be called elementary measures of the map T which satisfies (1.4).

Assume that the linear map $T\colon A \to L(H)$ satisfying (1. 4) is multiplicative, that is

(1. 7) $$T(uv) = T(u)T(v) \quad \text{for all} \quad u, v \in A.$$

We then call T a representation or more precisely an operator representation, of A.

If T is a representation of A then (1. 7) implies that $T(1)$ (1 stands here for the function identically equal to 1) is a projection. For every $u \in A$ and every $x \in H$ we have then $T(u)x = T(u)T(1)x$ and $T(u)(I-T(1))x = 0$. Hence, avoiding the trivial part $T(u)(I-T(1))$ of the representation T we can restrict it to $T(1)H$, which simply means that we can assume, as we do in the next section, that $T(1) = I$.

2. Suppose we are given the representation $T\colon A \to L(H)$ which satisfies (1. 4) and let $\{p_{x,y}\}$ be a certain collection of its elementary measures. We fix the Gleason part of $M(A)$ and decompose each $p_{x,y}$ as

$$p_{x,y} = p^a_{x,y} + p^s_{x,y},$$

where $p^a_{x,y} \ll G$ and $p^s_{x,y}$ is G-singular. We then obtain

(2. 1) $$(T(u)x, y) = \int u\, dp^a_{x,y} + \int u\, dp^s_{x,y} \qquad (u \in A,\ x, y \in H).$$

The equality $(T(u)(x+y), z) = (T(u)x, z) + (T(u)y, z)$ and (2. 1) imply that

$$\int u\, dp^a_{x+y,z} - \int u\, dp^a_{x,z} - \int u\, dp^a_{y,z} + \int u\, dp^s_{x+y,z} - \int u\, dp^s_{x,z} - \int u\, dp^s_{y,z} = 0,$$

that is

$$(p^a_{x+y,z} - p^a_{x,z} - p^a_{y,z}) + (p^s_{x+y,z} - p^s_{x,z} - p^s_{y,z}) \perp A.$$

By the M. and F. Riesz theorem (1.2) we have for $u \in A$

(2. 2) $$\int u\, dp^a_{x+y,z} = \int u\, dp^a_{x,z} + \int u\, dp^a_{y,z}, \quad \int u\, dp^s_{x+y,z} = \int u\, dp^s_{x,z} + \int u\, dp^s_{y,z}.$$

It follows that the functionals

$$\xi^a(u; x, y) = \int u\, dp^a_{x,y} \quad \text{and} \quad \xi^s(u; x, y) = \int u\, dp^s_{x,y}$$

are additive with respect to x. Using similar arguments, one verifies easily that these functionals are homogeneous in x and antilinear in y. Summing up we conclude that if u is fixed, ξ^a and ξ^s are bilinear forms on $H \times H$. On the other hand, $\|p^a_{x,y}\| \leq k\|x\|\,\|y\|$ and $\|p_s^{x,y}\| \leq k\|x\|\,\|y\|$, which implies that there are $T^a(u)$ and $T^s(u) \in L(H)$ such that for $x, y \in H$ we have $\xi^a(u; x, y) = (T^a(u)x, y)$, $\xi^s(u; x, y) = (T^s(u)x, y)$, and $\|T^a(u)\| \leq k\|u\|$, $\|T^s(u)\| \leq k\|u\|$. By (2. 1) and the definitions of ξ^a, ξ^s we have

(2. 3) $$T(u) = T^a(u) + T^s(u) \quad \text{for} \quad u \in A.$$

Since ξ^a, ξ^s are linear in u, $T^a(u)$ and $T^s(u)$ are linear in u. We did not use up till now the multiplicativity of T, which yields the equalities

$$(T(v)T(u)x, y) = \int uv\, dp^a_{x,y} + \int uv\, dp^s_{x,y} = (T(v)x, T(u)^*y) =$$
$$= \int v\, dp^a_{x,T(u)^*y} + \int v\, dp^s_{x,T(u)^*y}$$

which by (1. 2) prove that

$$\int v\, dp^a_{x,T(u)^*y} = \int uv\, dp^a_{x,y}, \quad \int v\, dp^s_{x,T(u)^*y} = \int uv\, dp^s_{x,y},$$

that is, by definition of T^a and of T^s,

(2. 4) $\quad (T^a(v)x, T(u)^*y) = (T^a(uv)x, y), \quad (T^s(v)x, T(u)^*y) = (T^s(uv)x, y).$

(2. 4) and (2. 3) give us

$$\int u\, dp^a_{T^a(v)x,y} - \int uv\, dp^a_{x,y} + \int u\, dp^s_{T^a(v)x,y} = 0.$$

Using (1. 2) we infer therefore that

$$(T^a(u)T^a(v)x, y) = \int u\, dp^a_{T^a(v)x,y} = \int uv\, dp^a_{x,y} = (T^a(uv)x, y).$$

Since x and y are arbitrary, the equality

(2. 5) $$T^a(uv) = T^a(u)T^a(v) \quad (u, v \in A)$$

follows. By the same token

(2. 6) $$T^s(uv) = T^s(u)T^s(v) \quad (u, v \in A),$$

which shows that both T^a and T^s are representations. We deduce therefore that $T^a(1)$ is a projection which together with the equality $T^a(1)+T^s(1) = I$ proves that (2. 3) gives us the decomposition of T into the direct sum of T^a and T^s.

The components T^a and T^s do not depend in some sense on the choice of elementary measures. Indeed, let $\{\tilde{p}_{x,y}\}$ be an arbitrary system of finite measures such that $(T(u)x, y) = \int u\, d\tilde{p}_{x,y}$ for all $u \in A$ and all $x, y \in H$. Applying (1. 2) and using the previous elementary measures $p_{x,y}$ we get that

$$(T^a(u)x, y) = \int u\, d\tilde{p}^a_{x,y} \quad \text{and} \quad (T^s(u)x, y) = \int u\, d\tilde{p}^s_{x,y}$$

for all $u \in A$ and all $x, y \in H$. This simply means that T^a and T^s do not depend on the manner we extend boundedly the functionals $u \to (T(u)x, y)$. All this gives rise to the following

Definition. We say that the linear bounded map $T: A \to L(H)$ is *G-continuous (G-singular)* if ther eis a system of finite measures $\{p_{x,y}\}$ such that $p_{x,y} \ll G$

($p_{x,y}$ G-singular) and $(T(u)x, y) = \int u\, dp_{x,y}$ for all $u \in A$ and all $x, y \in H$. We write $T \ll G$ if T is G-continuous and $T \perp G$ if T is G-singular.

Summing up we formulate the following theorem:

Theorem 2. 1. *Suppose T: $A \to L(H)$ is a linear map such that $\|T(u)\| \leqq k\|u\|$ for all $u \in A$. Then T may be written in a unique way as the sum $T = T^a + T^s$ of linear maps of A into $L(H)$ such that $T^a \ll G$ and $T^s \perp G$. The maps T^a, T^s satisfy $\|T^a(u)\| \leqq \leqq k\|u\|$ and $\|T^s(u)\| \leqq k\|u\|$ for $u \in A$. If T is a representation then both T^a and T^s are representations and T is the direct sum of T^a and T^s.*

Suppose we are given two Gleason parts, G_1 and G_2, of $M(A)$. We will prove the following

Theorem 2. 2. *Let T: $A \to L(H)$ be a representation of A. Let $T = T_i^a + T_i^s$ $(i=1, 2)$ be the decompositions of T such that $T_i^a \ll G_i$ and $T_i^s \perp G_i$ for $i=1, 2$ and assume that $G_1 \neq G_2$. Then*

$$T_1^a(u)\, T_2^a(v) = 0 \text{ and } T_1^s(u)\, T_2^a(v) = T_2^a(uv) \text{ for } u, v \in A.$$

Proof. Since $T = T_1^a + T_2^s = T_2^a + T_2^s$, the second part of Th. 2. 1 gives $TT_2^a = T_1^a T_2^a + T_1^s T_2^a$, which again by Th. 2. 1 implies that $(T_1^a(u)T_2^a(v)x, y) + (T_1^s(u)T_2^a(v)x, y) - (T_2^a(uv)x, y) = 0$. We get therefore for decompositions

$$p_{x,y} = p_{x,y}^{a,i} + p_{x,y}^{s,i} \qquad (p_{x,y}^{a,i} \ll G_i,\ p_{x,y}^{s,i}\ G_i\text{-singular})$$

of elementary measures $p_{x,y}$ of T the equalities

$$\int u\, dp^{a,1}_{T_2^a(v)x,y} - \int u\, dp^{s,1}_{T_2^a(v)x,y} - \int uv\, dp^{a,2}_{x,y} = 0,$$

which by (1. 2) gives us that

$$(T_1^a(u)\, T_2^a(v)x, y) = \int u\, dp^{a,1}_{T_2^a(v)x,y} = 0$$

and

$$(T_1^s(u)\, T_2^a(v)x, y) = \int u\, dp^{s,1}_{T_2^a(v)x,y} = \int uv\, dp^{a,2}_{x,y} = (T_2^a(uv)x, y),$$

because $p_{x,y}^{a,2}$ are G_1-singular for all x and y. Since x and y are arbitrary the proof is complete.

Using the notation of the above theorem we conclude that $T_2^a(v) = T_1^s(1)T_2^a(v)$ for $v \in A$, which implies the inclusion $T_2^a(A)H \subset T_1^s(1)H$. We can now decompose the representation T_1^s with respect to G_2 and thus obtain the decomposition of the original T into there parts, the first two of which are G_1- and G_2-continuous, respectively, and the last one is G_1- as well as G_2-singular. It is then possible to continue this approach using for instance the Kuratowski—Zorn lemma and thus get a full limit decomposition of T with regard to the totality of all Gleason parts of $M(A)$ plus a certain "completely singular" component, i.e. a G-singular one for each Gleason part G. In what follows we will be interested rather in the typical Hilbert space approach which leads to orthogonal decompositions. We assume namely in all what follows that

(2. 7) $$k=1, \quad \text{i.e.} \quad \|T(u)\| \leqq \|u\| \quad \text{for} \quad u \in A,$$

and once again explicitly that

(2. 8) $$T(1) = I.$$

The reformulation of Th. 2. 1 will read now as follows:

Theorem 2. 3. *Let $T: A \to L(H)$ be a representation which satisfies* (2. 7) *and* (2. 8). *Then T is a unique orthogonal sum $T = T^a \oplus T^s$ of representations $T^a \ll G$ and $T^s \perp G$.*

Let $\{G_\alpha\}$ be an indexed set of all Gleason parts of $M(A)$ such that if $\alpha \neq \beta$ then $G_\alpha \neq G_\beta$. Write $P_\alpha = T^a(1)$ where T^a is the G_α-continuous part of the representation T satisfying (2. 7) and (2. 8), in accordance to Th. 2. 3. It follows from Th. 2. 2 that $P_\alpha P_\beta = 0$ for $\alpha \neq \beta$. Certainly P_α are orthogonal projections. We define

$$P^a = \oplus P_\alpha \quad \text{and} \quad P^s = I - P^a.$$

Every subspace $H_\alpha = P_\alpha H$ reduces T. Consequently so does $P^s H$. It follows that

(2. 9) $$T(u) = (\oplus T(u) P_\alpha) \oplus T(u) P^s \quad \text{for all} \quad u \in A.$$

By Th. 2. 1 the part of T in every H_α is a representation. It follows that the part of T in $P^s H$ is a representation too.

The subspaces H_α and $H^s = P^s H$ may be characterized as follows:

(2. 10) The following conditions are equivalent:

(a) $x \in H_\alpha$ ($x \in H^s$).

(b) There exists a positive measure $p_0 \ll G_\alpha$ (p_0 completely singular) such that $(T(u)x, x) = \int u \, dp_0$ for all $u \in A$.

(c) If $\int u \, dp = (T(u)x, x)$ for each $u \in A$ for some positive measure p, then $p \ll G_\alpha$ (p is completely singular).

Proof. We will give the proof in the absolutely continuous case. The case of a completely singular part may be treated in a quite similar way. First notice that (a) implies (b) by the definition of H_α. To show that (b) implies (c) we argue as follows:

If $\int u \, dp = (T(u)x, x)$ for $u \in A$ then $p - p_0 \perp A$. Hence $(p - p_0^a) + p_0^s \perp A$, where $p^a \ll G$ and $p^s \perp G$. This gives us by (1. 2) that $\int u \, dp^s = 0$ for $u \in A$. Since $1 \in A$ and p^s is a positive measure, it vanishes identically. Q.E.D.

In order to prove that (c) implies (a) we proceed as follows:

Let $p_{x,x}$ be a suitable elementary measure. Assuming (c) we get that $p_{x,x} \ll G_\alpha$. Hence the part $p^s_{x,x} \perp G_\alpha$ vanishes. It follows that $0 = p^s_{x,x}(\Omega) = ((I-P_\alpha)x, x) = \|(I-P_\alpha)x\|^2$, which completes the proof.

Suppose now that besides of (2. 9) T has the decomposition

$$T(u) = (\oplus T_\alpha(u)) \oplus T_s(u),$$

where T_α are G_α-continuous representations and T_s is a completely singular one. Let

$$H = (\oplus H'_\alpha) \oplus H'_s$$

be the corresponding decomposition of the representation space. For $x \in H'_\alpha$ there is a measure $p \ll G_\alpha$ such that $(T(u)x, x) = (T_\alpha(u)x, x) = \int u\, dp$ for $u \in A$ because $T_\alpha \ll G_\alpha$. By (2. 10) $x \in H_\alpha$. It follows that $H'_\alpha \subset H_\alpha$. Hence $H_\alpha \ominus H'_\alpha \subset \left(\bigoplus_{\alpha \neq \beta} H'_\beta\right) \oplus H'_s$. Suppose now that $x \in H_\alpha$ and $x \perp H'_\alpha$. By the above inclusion there is a sequence $\{x_n\}$ such that x_0 is the orthogonal projection of x on H'_s and x_n is the orthogonal projection of x on a suitable subspace H'_{β_n}. Hence

$$(T(u)x, x) = \sum_{n=0}^{\infty} (T(u)x_n, x_n) \quad \text{for} \quad u \in A.$$

Since $x \in H_\alpha$, there is a positive measure $p \ll G_\alpha$ such that $(T(u)x, x) = \int u\, dp$ for $u \in A$. There are also a completely singular positive measure p_0 and positive measures $p_n \ll G_{\beta_n}$ $(n = 1, 2, \ldots)$ such that $(T(u)x_n, x_n) = \int u\, dp_n$ for $u \in A$ and $n = 0, 1, 2, \ldots$. It follows that $p - \sum_{n=0}^{\infty} p_n \perp A$. Since each p_n is G_α-singular $(\alpha \neq \beta_n)$, the measure $\tilde{p} = \sum_{n|0}^{\infty} p_n$ is G_α-singular. Hence the G_α-continuous part of $p - \tilde{p}$ equals p. By (1. 2) $p \perp A$, which implies that $(x, x) = \int 1 dp = 0$. This shows that $H_\alpha = H'_\alpha$. It is now a simple matter to show that the part of T in H_α equals T_α and the part in H^s equals T_s, which completes the proof of the uniqueness assertion of the following theorem:

Theorem 2. 4. *Suppose that $T: A \to L(H)$ is a representation of A which satisfies* (2. 7) *and* (2. 8). *Then T has a unique decomposition $T = (\oplus T_\alpha) \oplus T_s$ where T_α and T_s are representations of A such that $T_\alpha \ll G_\alpha$ for all α and T_s is completely singular.*

3. The G_α-continuous part T_α of the representation T admits an extension to a certain algebra consisting of limit functions of subsequences of A. The construction of such an extension is quite natural and simple in case of Ω-dilatable representations*). The related functional calculus for contractions may be found in [9] Chapt. III. In the general case considered in this paper we cannot apply the methods of dilation theory, because there is not known up till now whether every represent-ation is dilatable in a suitable way. In particular we cannot assert that

$$\|T_\alpha(u)x\|^2 \leqq \int |u|^2 \, dp_{x,x} \tag{3.1}$$

for some elementary measure $p_{x,x} \ll G_\alpha$ (provided $x \in H_\alpha$), which is the case when T and consequently T_α is Ω-dilatable. We shall overcome this difficulty in some way shown below.

To begin with we introduce the notation $H^\infty_{\alpha,0}$ for the class of Baire functions v such that $v = \lim v_n$ almost everywhere for each $m \in M_\varphi$ for some $\varphi \in G_\alpha$, where $v_n \in A$ and $\sup_n \|v_n\| < \infty$.

The results of [3] yield that if we identify functions which are equal up to G_α-null sets then $H^\infty_{\alpha,0}$ does not depend on the special choice of $\varphi \in G_\alpha$ (see (1.3)).

Since the construction of an extension of T_α does not require the multiplicativity of T we introduce for convenience the following condition:

($*$) The linear map $T\colon A \to L(H)$ satisfies (2.7) and (2.8), and for each $x \in H$ there is a positive measure p_x such that $p_x \ll G_\alpha$ and $(T(u)x, x) = \int u \, dp_x$ for $u \in A$.

Using the polarization formula and (1.2) one verifies easily that ($*$) is equivalent to the property that T has a system of G_α-continuous elementary measures.

Let us take now $v \in H^\infty_{\alpha,0}$ and let $v_n \to v$ a.e. for $m \in M_\varphi$ and $\|v_n\| \leqq K$ for some finite K. We take $x, y \in H$ and an elementary measure $p_{x,y} \ll G_\alpha$. Since $v_n \to v$ a.e. for $m \in M_\varphi$, the set on which $\limsup |v_n - v| > 0$ is of measure zero for each $m_\varphi \in M$. Consequently, by the dominated convergence theorem,

$$\lim (T(v_n)x, y) = \lim \int v_n \, dp_{x,y} = \int v \, dp_{x,y}.$$

The vectors x and y being arbitrary, there is an operator $\hat{T}(v)$ such that $\hat{T}(v) \in L(H)$ and $(\hat{T}(v)x, y) = \int v \, dp_{x,y}$ for $v \in H^\infty_{\alpha,0}$ and $x, y \in H$. Certainly $\hat{T}(v)$ does not depend on the choice of the approximating sequence $\{v_n\}$ but merely on v. It is a trivial matter to check that the map $\hat{T}\colon H^\infty_{\alpha,0} \to L(H)$ is linear. All the above holds true under the assumption ($*$). Write now

$$\|v\|_{\infty,0} = \inf_{\sigma \in N} \{\sup_\sigma |v|\} \quad (v \in H^\infty_{\alpha,0}),$$

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*) For the definition of Ω-dilatability and related matters see [7].

where N stands for the totality of all φ-null sets. Assuming ($\ast$) we infer that

$$|(\hat{T}(v)x, y)| \leqq \int |v| \, d|p_{x,y}| \leqq \|v\|_{\infty,0} \|x\| \, \|y\|$$

(because $\|p_{x,y}\| \leqq \|x\| \, \|y\|$). Hence $\hat{T}$ is a well-defined linear map of the quotient Banach algebra $H_\alpha^\infty = H_{\alpha,0}^\infty / S$ where $S = \{v : \|v\|_{\infty,0} = 0\}$, endowed with the norm $\|v\|_\infty$ induced by $\|\cdot\|_{\infty,0}$. We have

$$\|\hat{T}(v)\| \leqq \|v\|_\infty \quad \text{for} \quad v \in H_\alpha^\infty. \tag{3.2}$$

We will show next that $\hat{T}$ is a representation of H_α^∞, provided T itself is a representation and ($\ast$) holds true.

Let us consider the Banach space

$$E = L^1(|p_{x,y}|) \times L^1(|p_{x,y}|) \times H \times H$$

with the norm

$$\|\{u, v, z_1, z_2\}\| = \|u\|_{L^1} + \|v\|_{L^1} + \|z_1\| + \|z_2\|$$

and suppose that $u_n \to u \in H_\alpha^\infty$, $v_n \to v \in H_\alpha^\infty$ a.e. for $m \in M_\varphi$, where $u_n, v_n \in A$ and $\sup_n \{\|v_n\| + \|u_n\|\} < +\infty$. Then

$$\{u_n, v_n\} \to \{u, v\} \quad \text{strongly in} \quad L^1(|p_{x,y}|) \times L^1(|p_{x,y}|).$$

On the other hand

$$(T(u_n)z_1, z_2) \to (\hat{T}(u)z_1, z_2), \quad (T(v_n)^* z_1, z_2) \to (\hat{T}(v)^* z_1, z_2)$$

for all z_1 and z_2. It follows that

$$\{u_n, v_n, T(u_n)x, T(v_n)^* y\} \to \{u, v, \hat{T}(u)x, \hat{T}(v)^* y\} = q$$

weakly in E. By the classical theorem of S. MAZUR there exists a sequence $\lambda_{i,n} \geqq 0$ ($i = 1, 2, \ldots, n$; $n = 1, 2, 3, \ldots$) such that $\sum_{i=1}^n \lambda_{i,n} = 1$ for all n and such that for

$$\hat{u}_n = \sum_{i=1}^n \lambda_{i,n} u_i, \quad \hat{v}_n = \sum_{i=1}^n \lambda_{i,n} v_i, \quad x_n = T(\hat{u}_n)x, \; y_n = T(\hat{v}_n)^* y$$

we have

$$\{\hat{u}_n, \hat{v}_n, x_n, y_n\} \to q \quad \text{strongly in} \quad E.$$

Consequently

$$(x_n, y_n) = (T(\hat{u}_n)x, T(\hat{v}_n)^* y) = (T(\hat{u}_n \hat{v}_n)x, y) = \int \hat{u}_n \hat{v}_n \, dp_{x,y} \to$$

$$\to \int uv \, dp_{x,y} = (\hat{T}(uv)x, y).$$

The strong convergence yields that

$$(T(\hat{u}_n)x, T(\hat{v}_n)^* y) \to (\hat{T}(u)x, \hat{T}(v)^* y)$$

which shows now that $(\hat{T}(u)\hat{T}(v)x, y) = (\hat{T}(u)x, \hat{T}(v)^* y) = (\hat{T}(uv)x, y)$. Since x and y are arbitrary, $\hat{T}(uv) = \hat{T}(u)\hat{T}(v)$ for $u, v \in H^*$. Q.E.D.

Assuming only $(\ast)$ we get that $\hat{T}$ is *weakly continuous* in the following sense: If $v_n, v \in H_\alpha^\infty$, $\sup \|v_n\|_\infty < +\infty$, and $v_n \to v$ a.e. for each $m \in M_\varphi$, then $\hat{T}(v_n) \to \hat{T}(v)$ weakly.

It is a trivial matter to show that if some weakly continuous linear bounded map coincides on A with $\hat{T}$, then it is identical with $\hat{T}$. Summing up we get the following theorem:

Theorem 3. 1. *Suppose that $(\ast)$ holds true. Then there is a unique weakly continuous linear map $\hat{T}: H_\alpha^\infty \to L(H)$ such that $T(u) = \hat{T}(u)$ for $u \in A$. Moreover, $\|\hat{T}(u)\| \leqq \|u\|_\infty$ for $u \in H_\alpha^\infty$. If the original T is a representation of A then $\hat{T}$ is a representation of H_α^∞.*

If T is Ω-dilatable then (3. 1) holds true. This implies that $\hat{T}$ is then strongly continuous. More precisely, the following holds true:

(3. 3) If T satisfies $(\ast)$ and is Ω-dilatable, and if $v_n, v \in H_\alpha^\infty$, $\sup \|v_n\|_\infty < \infty$, and $v_n \to v$ a.e. for $m \in M_\varphi$, then $\hat{T}(v_n) \to \hat{T}(v)$ strongly.

If φ admits central measures, i.e. measures $m' \in M_\varphi$ such that $m \ll m'$ for all $m \in M_\varphi$, then H_α^∞ may be treated as the Banach subalgebra of $L^\infty(m')$. This is the case if Ω is a metric space (see [9]). If A is a hypo-dirichlet algebra then the (necessarily unique) Arens—Singer measure $m' \in M_\varphi$ is central (see [1]), and moreover

$$H_\alpha^\infty = L^\infty(m') \cap H^2(m') \tag{3. 4}$$

where $H^2(m')$ is the $L^2(m')$ closure of A ([1] Col. p. 129). If M_φ reduces to a single measure set $\{m\}$ then (3. 4) holds true on the basis of the extension of the Wermer—Hoffman lemma given in [3] (Th. 2. 1 of [3]). In this case our Th. 3. 1 gives immediately the functional calculus of contractions having unitary dilations with Lebesgue spectrum. The algebra H_α^∞ is then simply the disc algebra H^∞. Suppose A separates the points of Ω.

Theorem 3. 2. *Assume that the linear map T satisfies $(\ast)$. Suppose that there is a unique probability measure m representing $\varphi \in G_\alpha$, where G_α is the Gleason part of $M(A)$. Then there exists a unique semi-spectral measure $F: B \to L(H)$ (on the σ-field B of Baire sets in Ω) such that*

$$T(u) = \int u\, dF \quad \textit{for} \quad u \in A.$$

F is absolutely continuous with regard to m, i.e. $(Fx, x) \ll m$ for every $x \in H$.

The proof of the above theorem is based on the following property:

(3. 5) Suppose the homomorphism φ has a unique representing probability measure m. If $\int u h\, dm = 0$ for some real $h \in L^1(m)$ and all $u \in A$, then $h = 0$ a.e. for m.

The proof of (3.5) is exactly the same as that of Th. 6. 7 of [4]. One has to use Th. 4 of [6] which together with the Arens lemma quoted in [4] (Lemma 6. 6) guarantees that the arguments applied in [4] work well under the only assumption of the uniqueness of m for a single φ.

Proof of Theorem 3. 2. It follows from (3. 5) that for every $z \in H$ there is a unique positive measure $p_z \ll m$ such that

$$(T(u)z, z) = \int u\, dp_z \quad \text{for} \quad u \in A.$$

Since for $x, y \in H$

$$T((u)(x+y), x+y) + (T(u)(x-y), x-y) = 2\{(T(u)x, x) + (T(u)y, y)\},$$

we have $p_{x+y} + p_{x-y} - 2[p_x + p_y] \perp A$, which by (3. 5) implies

$$p_{x+y} + p_{x-y} = 2\{p_x + p_y\}. \tag{3.6}$$

Next, since $\int u\, dp_{\alpha x} = (T(u)\alpha x, \alpha x) = |x|^2 \int u\, dp_x$, we get by similar arguments

$$p_{\alpha x} = |\alpha|^2 p_x. \tag{3.7}$$

Define now $p_{x,y} = \frac{1}{2}(p_{x+y} - p_{x-y})$. Then for real α

$$\int u\, dp_{\alpha x, y} = \frac{1}{4}\{(T(u)(\alpha x + y), \alpha x + y) - (T(u)(\alpha x - y), \alpha x - y)\} =$$

$$= \frac{1}{4}\{(T(u)(x+y), x+y) - (T(u)(x-y), x-y)\} = \alpha \int u\, dp_{x,y}$$

which by (3. 5) gives

$$p_{\alpha x, y} = \alpha p_{x,y} \quad \text{for real} \quad \alpha. \tag{3.8}$$

Using now the suitable parts of the proof of Th. I of [10] we infer from (3. 6)—(3. 8) that, setting $p_{x,y} = p_{x,y} - p_{ix,y}$, $q_{x,y}(\sigma)$ is for each fixed Baire set σ a hermitian symmetric bilinear form in x and y such that $q_{x,x} = p_x$ for every x. Hence $\|q_{x,x}\| = \|x\|^2$, which implies that there is a semi-spectral measure F such that $T(u) = \int u\, dF$ for all $u \in A$. If $T(u) = \int u\, dE$ for some other semi-spectral E then by (1. 2) $E \ll m$. It follows then that $A \perp (Fx, x) - (Ex, x) \ll m$, which by (3. 5) gives that $(Ex, x) = (Fx, x)$. Q.E.D.

Theorem 3. 2 is equivalent to the following statement: Under the uniqueness of the representing measure m of $\varphi \in M(A)$, every map T satisfying ($*$) with $\varphi \in G_\alpha$ is Ω-dilatable in an essentially unique way.

Assume for a while that every $\varphi \in M(A)$ has a unique representing measure and let T be a representation of A satisfying (2. 7) and (2. 8). It follows then from Th. 2. 3 and from the above that the part $\oplus T(u)P_\alpha$ of decomposition (2. 9) is uniquely Ω-dilatable.

Combining Th. 3. 1 with Th. 3. 2 and Remark (3. 3) we get the following

Theorem 3. 3. *Suppose m is the unique representing measure for $\varphi \in M(A)$. Let T be a linear map satisfying $(\ast)$. Then there exists a unique strongly continuous map $\hat{T}$ of $H_\alpha^\infty(m) = L^\infty(m) \cap H^2(m)$ into $L(H)$ such that $T(u) = \hat{T}(u)$ for $u \in A$. Moreover, $\|\hat{T}(u)\| \leqq \|u\|_\infty$ for $u \in H_\alpha^\infty$ and $\hat{T}(u) = \int u\, dF$ for $u \in H_\alpha^\infty$ where F is the semi-spectral measure corresponding to T by Th. 3. 2. If T is a representation then so is $\hat{T}$.*

It follows from Th. 4 of [5] that the uniqueness of m implies that $H_\alpha^\infty(m)$ is a logmodular algebra on the maximal ideal space $\hat{\Omega}$ of the algebra $L^\infty(m)$. Using other results of [4] and [5], namely the extended Gleason—Whitney extension theorem, one verifies that

$$\hat{T}(u) = \int \hat{u}\, d\hat{F} \quad \text{for} \quad \hat{u} \in H_\alpha^\infty(\hat{m}) \tag{3.10}$$

where $\hat{u}$ is the Gelfand image on $\hat{\Omega}$ of $u \in L^\infty(m)$, $\hat{m}$ stands for the unique representing measure on $\hat{\Omega}$ of the positive extension on $H_\alpha^\infty(m)$ of the functional $u \to \int u\, dm$ $(u \in A)$, and $\hat{F}$ is the $\hat{m}$-continuous semi-spectral measure on Baire subsets of $\hat{\Omega}$. The uniqueness of $\hat{F}$ follows from Th. 3. 2 by the $\hat{m}$-continuity of $\hat{F}$ by putting $A = H_\alpha^\infty(\hat{m}) \subset C(\hat{\Omega})$. Consequently, if the original T is a representation of A then $\hat{T}$: $A = H_\alpha^\infty(\hat{m}) \to L(H)$ is a G-continuous representation, where G is the Gleason part of $M(H_\alpha^\infty(\hat{m}))$ which includes $\hat{m}$, and the decomposition (2. 9) reduces to the G-continuous component.

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